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DEVELOPED MOTION OF ROBOT END-EFFECTOR OF TIMELIKE RULED SURFACES WITH SPACELIKE RULINGS (THE FIRST CASE)

Gülnur Şaffak Atalay, Emin Kasap

Abstract
The trajectory of a robot end effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. In this paper, we analyzed the problem of describing trajectory of a robot end-effector by a timelike ruled surface with spacelike ruling. We obtained the developed frame $\{k_1(s), r_1(s), t_1(s)\}$ by rotating the generator frame $\{r(s), t(s), k(s)\}$ at an Darboux angle $\theta = \theta(s)$ in the plane $\{r(s), k(s)\}$, which is on the striction curve $\beta(s)$ of the timelike ruled surface . Afterword, natural frame, tool frame and surface frame which is necessary for the movements of robot are defined derivative formulas of the frames are founded by calculating the Darboux vectors. Tool frame $\{O(s), A(s), N(s)\}$ are constituted by means of this developed frame. Thus, robot end effector motion is defined for the timelike ruled surface φ generated by the orientation vector $k_1(s) = O(s)$. Also, by using Lancret curvature of the surface and rotation angle (Darboux angle) in the developed frame the robot end-effector motion is developed. Therefore, differential properties and movements an different surfaces in Minkowski space is analyzed by getting the relations for curvature functions which are characterized a timelike ruled surface with spacelike directix. Finally, to be able to get a member of trajectory surface family which has the same trajectory curve is shown with the examples in every different choice of the Darboux angle which is used to described the developed frame.

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Ondokuz Mayıs University, Faculty of Education, Department of Mathematics Education, Samsun, Turkey,Ondokuz Mayıs University, Faculty of Arts and Sciences, Department of Mathematics, Samsun, Turkey
Gülnur Şaffak Atalay: gulnur.saffak@omu.edu.tr
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1. Introduction

The motion of a robot end effector is referred to as the robot trajectory. The trajectory of a robot end effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. A robot trajeectory consist of; (i) a sequence of positions, velocities and accelerations of a point fixed in the end effector, and (ii) a sequence of orientations, angular velocities and angular accelerations of the end effector. The point fixed in the end effector will be referred to as the Tool Center Point and denoted as the TCP. Ruled surfaces were first investigated by G. Monge who established the partial differential equation satisfied by all ruled surfaces. Ruled surfaces have been widely applied in designing cars, ships, manufacturing (e.g. , CAD/CAM) of products and many other areas such as motion analysis and simulation of rigid body, as well as model-based object recognition system. However, ruled surfaces are stil widely used in many areas in modern surface modelling systems. Ruled surfaces in Minkowski 3-space have been studied in a lot of fields. More information about timelike ruled surfaces in Minkowski 3-space may be also found in Turgut and Hacısalihoğlu's papers in [1-2] and Öğrenmiş et al, [3]. Curvature theory investigates the intrinsic geometric properties of the trajectory of points, lines, and planes embedded in a moving rigid body. Curvature theory is also concerned with the velocity and acceleration distribution of a moving rigid body in constrained motion. The curvature theory is using to determine the differential properties of the motion of a robot end effector. The differential properties of the robot end effector motion are then related to the time dependent properties of the motion which are essential in the robot trajectory planning. The differential properties of the ruled surface generate the linear and angular motion properties of the robot end effector for robot path planning, [4]. Also, the curvature theory of line trajectories seeks to characterize the shape of the trajectory ruled surface and relates it to the motion of body carrying the linet hat generates it,[5]. Ryuh and Pennock, [6] applied the curvature theory of a ruled surface to study the instantaneous motion properties of a robotic device. The differential properties of motion of the end effector were determined from the curvature theory. Also, they proposed a method of robot trajectory planning based on the curvature theory of a ruled surface incorporated with the geometric modeling technique in [4]. In this method, it is shown that how a ruled surface may be generated using the geometric modeling technique of a curve. Ryuh, Lee and Moon in [7] studied a precision control method of a robot path generation based on the dual curvature theory a ruled surface. In [8], the authors are developed a new adjustment method for improving machining accuracy of tool path in five-axis flank milling of ruled surfaces. They proposed a feedrate adjustment rule that automatically controls the tool motion at feedrate-sensitive corners based on a bisection method. Also they are conducted on different ruled surfaces to verify the effectiveness of the proposed method. Kim et al. [9] developed a real-time trajectory generation method and control approach for a five-axis NC machine. They describe the spatial trajectory of the tool of the five axis machine by a ruled surface, and the differential motion parameters of the tool were obtained from the curvature theory of the ruled surface. Also, they were used the Fergusen geometric modeling technique to present the tool trajectory as a ruled surface. Also, in [10-12] the authors have studied manipulators. The motion of robot end-effector is a research topic of various studies in Minkowski 3-space. Ekici et al. studied the differential properties of robot end-effectors motion using the curvature theory of timelike ruled surfaces with timelike ruling in [13]. In [14], Turhan and Ayyıldız used the curvature theory of ruled surfaces with lightlike ruling in Minkowski 3-space. They also derived the relation between these functions and the curvature functions of the central normal surface whose ruling spacelike. Recently, in [15], Sahiner et al. studied the motion of a robot end-effector by using the curvature theory of a dual unit hyperbolic spherical curve which corresponds to a timelike ruled surface with timelike ruling generated by a line fixed in the end-effector. In this way, the linear and angular differential properties of the motion of a robot end-effector such as velocities and accelerations which are important information in robot trajectory planning are determined. They also derived the motion of a robot end-effector which moves on the surface of a right circular hyperboloid of one sheet is examined as a practical example. In this paper, we address the path planning problem using the curvature theory of a ruled surface. The objects consist of point in the coordinate plane. We can locate such coordinates by rotating these objects in a specific direction. This allows the calculation of the robot's next motion. So, any errors and miscalculations that may arise in trajectory planning can be prevented. Each robot has a unique coordinate system. However, the appropriate choice of coordinates for the robot motion allows us to define the work area of the robot more efficiently. Therefore, we obtained the developed frame $\{k_1(s), r_1(s), t_1(s)\}$ by rotating the generator frame $\{r(s), k(s)\}$ at an angle $\theta = \theta(s)$ in the plane $\{r(s), k(s)\}$ to provide a practical work area. It is useful in animation motion planning, and tool path planning in CAD/CAM. Thus, this study represents robot path as a ruled surface generated at the Tool Center Point and by the unit vector ($k_1(s) = O(s)$) of the tool frame. The vector $k_1(s) = O(s)$ is depending on the Darboux angle function $\theta = \theta(s)$ .New direction vectors are achieved by changing the angle function. The robot trajectory changes depending on the angle function. Therefore, we obtained trajectory ruled surface family with a common trajectory curve in the rotation trihedron. Any other generated trajectory corresponds to a member of this trajectory ruled surface family. The given calculations (i.e, positional variation of the TCP, linear velocity, angular velocity) are valid for all members of the trajectory ruled surface family. Therefore, we defined the desired trajectory of the robot end effector motion and give the differential properties of robot end effectors motion using the curvature theory. Also, the motion of robot end effector is illustrated with examples by two members of the trajectory ruled surface family.

2. Preliminaries

Let us consider Minkowski 3-Space $\mathbb{R}^3_1$ and let the Lorentzian inner product of $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ be

$$g(X,Y) = -x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

The norm of a vector X is defined by $\|X\| = \sqrt{|g(X,X)|}$ [16]. A vector $X = (x_1, x_2, x_3) \in \mathbb{R}^3_1$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ or $X = 0$ and lightlike (or null) if $g(X,X) = 0$ and $X \neq 0$ [16]. A timelike vector is to be positive (resp.negative) if and only if $x_3 > 0$ (resp. $x_3 < 0$) The vector product of vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ is defined by

$$X \times Y = (x_2 \cdot y_3 - x_3 \cdot y_1, x_1 \cdot y_3 - x_3 \cdot y_1, x_2 \cdot y_1 - x_1 \cdot y_2).$$

The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ are Lorentzian orthogonal if and only if $g(X,X) = 0$ [17].

Lemma 1. *Let X and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}^3_1$. If X is timelike, then Y is spacelike [17].*

Lemma 2. *Let X and Y be positive (negative) timelike vectors in* $\mathbb{R}^3_1$. *Then*

$$g(X,Y) \le \|X\| \, \|Y\|$$

whit equality if and only if X and Y are linearly dependent [17].

Lemma 3. ***i)*** *Let X and Y be positive (negative) timelike vectors in* $\mathbb{R}^3_1$*. By Lemma 2, there is a unique nonnegative real number* $\varphi(X,Y)$ *such that*

$$g(X,Y) = \|X\| \, \|Y\| \cosh \varphi(X,Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$ [17].
ii) *Let X and Y be spacelike vectors in* $\mathbb{R}^3_1$ *that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \le \|X\| \, \|Y\|.$$

Hence, there is a unique real number $\varphi(X,Y)$ *between* 0 *and* π *such that*

$$g(X,Y) = \|X\| \, \|Y\| \cos \varphi(X,Y).$$

$\varphi(X,Y)$ *is defined to be the Lorentzian spacelike angle between X and Y [17].*
iii) *Let X and Y be spacelike vectors in* $\mathbb{R}^3_1$ *that span a timelike vector subspace. Then, we have*

$$g(X,Y) > \|X\| \, \|Y\|.$$

Hence, there is a unique positive real number $\varphi(X,Y)$ *between* 0 *and* π *such that*

$$|g(X,Y)| = \|X\| \, \|Y\| \cosh \varphi(X,Y).$$

$\varphi(X,Y)$ *is defined to be the Lorentzian timelike angle between X and Y [17].*
iv) *Let X be a spacelike vector and Y be a positive timelike vector in* $\mathbb{R}^3_1$*. Then there is a unique nonnegative real number* $\varphi(X,Y)$ *such that*

$$|g(X,Y)| = \|X\| \, \|Y\| \sinh \varphi(X,Y).$$

$\varphi(X,Y)$ *is defined to be the Lorentzian timelike angle between X and Y [17].*

Theorem 4. *Let X, Y in* $\mathbb{R}^3_1$ *. We have*
i) If X and Y are the spacelike vectors , $X \times Y$ *is a timelike vector*
ii) If X and Y are the timelike vectors, $X \times Y$ *is a spacelike vector*
iii) If X is the spacelike vector and Y is the timelike vector, $X \times Y$ *is a spacelike vector*
where, $\times$ *is Lorentzian cross product , [1-2].*

- Let $\alpha = \alpha(s)$ be a unit speed curve in $\mathbb{R}^3_1$;by $\kappa = \kappa(s)$ and torsion $\tau = \tau(s)$ we denote the natural curvature and torsion of $\alpha(s)$,respectively. $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α, where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve α, respectively.

- Let α be a unit speed timelike curve with curvature κ and torsion τ. So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

  $$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

  where $\times$ is the Lorentzian cross product in $\mathbb{R}^3_1$ [18]. The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}^3_1$. Then, Frenet formulas are given by [18]

  $$T' = \kappa N, \; N' = \kappa T + \tau B, \; B' = -\tau N.$$

- Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation, we have

  $$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

  The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [18]

  $$T' = \kappa N, \; N' = \kappa T + \tau B, \; B' = \tau N.$$

- Let α be a unit speed spacelike curve with timelike binormal. In this case, T and N are spacelike vector fields and B is a timelike vector field and we have the following vectoral relation

  $$T \times N = B, \quad N \times B = -T, \quad B \times T = -N,$$

  The binormal vector field $B(s)$ is the unique timelike unit vector field perpendicular to the spacelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as $\mathbb{R}_1^3$. Then, Frenet formulas are given by [18]

  $$T' = \kappa N, \; N' = -\kappa T + \tau B, \; B' = \tau N.$$

A surface M in $\mathbb{R}_1^3$ is called a timelike surface if the induceded metric on the surface is a Lorentz metric [14]. The normal vector on the timelike surface is a spacelike vector , [16]. A timelike ruled surface in $\mathbb{R}_1^3$ is obtained by a timelike straight line moving along spacelike curve or by a spacelike straight line moving along a timelike curve, [16]. The timelike ruled surface M is given parameterization
$\Psi : I \times IR \to IR_1^3$, $\Psi(s, v) = \alpha(s) + vX(s)$ in $\mathbb{R}_1^3$.

3. Frames of Reference

A timelike ruled surface which indicates the tool path has a parametric representation,

$$X(s, v) = \alpha(s) + v\overline{R}(s) \tag{3.1}$$

where $\alpha(s)$ a timelike curve is the specified path of the TCP, v is a real-valued parameter and $\overline{R}(s)$ spacelike straigth line is the vector generating the timelike ruked surface (called the ruling). The striction curve of the timelike ruled surface X is

$$\beta(s) = \alpha(s) - \mu(s)\overline{R}(s) \tag{3.2}$$

where the parameter

$$\mu(s) = \left\langle \alpha'(s), \overline{R}'(s) \right\rangle \tag{3.3}$$

indicates the distance from the striction point of the timelike ruled surfaces to the TCP.

For the generator trihedron, there are two cases. The generator trihedron is defined by three mutually orthogonal unit vectors, namely;

i) the spacelike generator vector $r = (1/R)\overline{R}(s)$, the timelike central normal vector $t = \overline{R}'(s)$ and the spacelike central tangent vector $k = t \times r$, where R is $\left\|\overline{R}(s)\right\|$.

ii) the spacelike generator vector $r = (1/R)\overline{R}(s)$, the spacelike central normal vector $t = \overline{R}'(s)$ and the timelike central tangent vector $k = r \times t$, where R is $\left\|\overline{R}(s)\right\|$.

Now let's make the ecessary calculations for the first of these cases. Likewise it can be done in the other. The first order positional variation of the striction line of the timelike ruled surface may be expressed in the generator trihedron as

$$\beta' = \Gamma r + \Delta k \tag{3.4}$$

where

$$\begin{cases} \Gamma = \frac{1}{R}\left\langle \alpha', \overline{R} \right\rangle - \mu' R \\ \Delta = \frac{1}{R}\left\langle \alpha', \overline{R}' \times \overline{R} \right\rangle \end{cases} \tag{3.5}$$

is referred to as the curvature functions of the timelike ruled surface.
First order angular variation of the generator trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{pmatrix} r \\ t \\ k \end{pmatrix} = \frac{1}{R}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\gamma \\ 0 & -\gamma & 0 \end{pmatrix}\begin{pmatrix} r \\ t \\ k \end{pmatrix} \tag{3.6}$$

where
$$\gamma = \left\langle \overline{R}'', \overline{R}\times\overline{R}' \right\rangle$$
is referred to as the geodesic curvature of the curve drawn by the ruling vectors $\overline{R}(s)$ of the timelike ruled surface.
For the Darboux vector of generator trihedron of timelike ruled surface, we can write

$$U_r = t \times t' = \frac{1}{R}(\gamma r + k) \tag{3.7}$$

Also, the Lancret curvature of timelike ruled surface X is

$$\lambda = \|t'\| = \sqrt{\left|\frac{\gamma^2+1}{R^2}\right|} \tag{3.8}$$

4. Developed Trihedron

Let us rotate the generator trihedron $\{r,t,k\}$ on the striction curve of the timelike ruled surface X at the central point at an Darboux Lorentzian angle $\theta = \theta(s)$, $\theta \neq$fixed, in the plane $\{r,k\}$. So, it can be written in matrix form as

$$\begin{pmatrix} r_1 \\ k_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r \\ k \end{pmatrix} \tag{4.1}$$

In fact, by using the above rotation equation and we have relations

$$\begin{aligned} r_1 &= \cos\theta r - \sin\theta k, \\ t_1 &= t, \\ k_1 &= \sin\theta r + \cos\theta k. \end{aligned}$$

The orthonormal system $\{k_1 = U, r_1, t_1\}$ is called the developed trihedron of the timelike ruled surface X. Here, k_1 is Darboux vector of generator trihedron of the timelike ruled surface X.
Eqns.(3.7) and (3.9) may be written as

$$\sin\theta - \gamma\cos\theta = 0 \tag{4.2}$$

By using Eqn. (4.2) into Eqn.(3.8) we get the relations

$$\begin{cases} \frac{1}{R} = \mp\lambda\cos\theta \\ \frac{\gamma}{R} = \mp\lambda\sin\theta \end{cases} \tag{4.3}$$

The first-order angular variation of developed trihedron $\{k_1, r_1, t_1\}$ may be expressed in the matrix form as

$$\frac{d}{ds}\begin{pmatrix} k_1 \\ r_1 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 & \theta' & 0 \\ -\theta' & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ r_1 \\ t_1 \end{pmatrix} \tag{4.4}$$

where θ' is the curvature and λ is the Lancret curvature of the timelike ruled surface X. Also; k_1,r_1 and t_1 be the spacelike generator vector, the spacelike central normal vector and the timelike central tangent vector, respectively.

Each vector of developed trihedron in end effector defines its own ruled surface while the robot moves. Let us take the following timelike ruled surface (robot trajectory ruled surface) as

$$\varphi(s,v) = \alpha(s) + vk_1(s) \tag{4.5}$$

where $\alpha(s)$ timelike curve is the specified path of the TCP (called the directrix of the timelike ruled surface X and φ), v is a real valued parameter and $k_1(s)$ is the spacelike vector generating the timelike ruled surface φ (called the ruling or direction). Also, this surface is trajectory surface of robot.

If you take a surface formed by a t_1 timelike central tangent vector , such a surface is not defined in IR_1^3 . So, is not talk about the robot trajectory ruled surface.

If you take a surface formed by a r_1 spacelike central normal vector , such a surface is to be central normal surface, will examine in section 5.

The striction curve of timelike ruled surface φ is

$$\beta_{k_1}(s) = \alpha(s) - \mu_{k_1}(s)k_1(s) \tag{4.6}$$

where the parameter

$$\mu_{k_1}(s) = \frac{\Gamma\cos\theta - \Delta\sin\theta + \mu' R\cos\theta}{\theta'} \tag{4.7}$$

Also, Γ and Δ are referred to as the curvature functions of the timelike ruled surface (Eqn. (3.1)). Differentiating Eqn.(4.6) gives first order positional variation of the striction point of the timelike ruled surface φ. By using Eqns. (4.4) and (4.7) we can write Eqn.(4.6) with respect to developed trihedron as

$$\beta'_{k_1}(s) = \Gamma_{k_1}k_1 + \mu t_1 \tag{4.8}$$

where

$$\Gamma_{k_1} = \Gamma\sin\theta + \Delta\cos\theta + \mu' R\sin\theta - \mu'_{k_1} \tag{4.9}$$

How that, Γ and Δ are curvature functions which characterize the timelike ruled surface here Γ_{k_1} and μ are curvature functions which characterize the robot trajectory timelike ruled surface.

The positional variation of the striction line may be considered as the linear velocity. As in the case of the developed frame (4.4) may also be written as

$$U_{k_1} = r_1 \times r'_1 = \theta' t_1 - \lambda k_1 \tag{4.10}$$

which is Darboux vector of the developed frame. In a planar curve, the first term will drop out and the developed frame will rotate around the k_1 vector with an angular velocity. This formulation is useful for studying the rotational motions of rigid body attached to the developed frame moving along a curve.

5. Central Normal Surface

As the developed trihedron moves along the striction curve , the central normal vector generates another timelike ruled surface which is called the timelike central normal surface. The timelike central normal surface is defined as

$$\varphi_{r_1}(s,v) = \beta_{k_1}(s) + vr_1(s) \tag{5.1}$$

The striction curve of timelike central normal surface is

$$\beta_{r_1}(s,v) = \beta_{k_1}(s) - \mu_{k_1}(s)r_1(s) \tag{5.2}$$

Differentiating (5.2) and by using Eqn. (4.8) into Eqn. (4.4) gives

$$\mu_{r_1}(s) = \frac{\Gamma_{k_1}\theta' + \lambda\mu}{\theta'^2 - \lambda^2} \tag{5.3}$$

The natural trihedron is defined by the following three orthonormal vectors; the timelike central normal vector , spacelike principal normal vector , and spacelike binormal vector, as shown in figure 1. Also, the natural trihedron is used to study the angular and positional variation of the normal vector.

For the natural trihedron $\{r_1, r_2, r_3\}$, there are two cases:

i) r_2 timelike vector, r_2 and r_3 spacelike generator vector:

These three vectors are defined, respectively, as

$$\begin{cases} r_1 = \frac{k_1'}{\theta'} \\ r_2 = \frac{1}{\kappa} r_1' \\ r_3 = r_2 \times r_1 \end{cases} \tag{5.4}$$

where $\kappa = \|r_1'\|$ is the curvature of the timelike ruled surface φ . Also, here is $r_1 \times r_2 = -r_3$, $r_2 \times r_3 = -r_1$ and $r_3 \times r_1 = r_2$.

ii) r_3 timelike vector, r_1 and r_2 spacelike generator vector:

These three vectors are defined, respectively, as

$$\begin{cases} r_1 = \frac{k_1'}{\theta'} \\ r_2 = \frac{1}{\kappa} r_1' \\ r_3 = r_1 \times r_2 \end{cases}$$

where $\kappa = \|r_1'\|$ is the curvature of the timelike ruled surface φ . Also, here is $r_1 \times r_2 = r_3$, $r_2 \times r_3 = -r_1$ and $r_3 \times r_1 = -r_2$.

Now let's make the necessary calculations for the first of these cases. Likewise it can be done in the other.

Let η be the angle between the spacelike vectors k_1 and r_3, see figure 1. Here, the developed trihedron and the natural trihedron have the timelike central normal vector in common. Then, we have

$$\begin{cases} k_1 = \sinh\eta\, r_2 + \cosh\eta\, r_3 \\ t_1 = \cosh\eta\, r_2 + \sinh\eta\, r_3 \end{cases} \tag{5.5}$$

Substituting Eqn. (4.4) into Eqn. (5.5) and using $r_1' = \kappa r_2$ it follows that

$$\begin{cases} \cosh\eta = \frac{\lambda}{\kappa} \\ \sinh\eta = \frac{\theta'}{\kappa} \end{cases} \tag{5.6}$$

From Eqn. (5.6), adding the result and rearranging gives the curvature

$$\kappa = \sqrt{\lambda^2 - \theta'^2} \tag{5.7}$$

The Darboux vector of developed trihedron may be obtained in the natural trihedron as follows. Substituting Eqn. (5.6) into Eqn. (5.5) gives

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{\kappa} \begin{pmatrix} 0 & \kappa & 0 \\ -\theta' & 0 & \lambda \\ \lambda & 0 & -\theta' \end{pmatrix} \begin{pmatrix} k_1 \\ r_1 \\ t_1 \end{pmatrix} \tag{5.8}$$

Hence, the Darboux vector of the developed trihedron may be written as

$$U_{k_1} = -\kappa r_3 \tag{5.9}$$

which shows that the binormal vector plays the role of the opposite direction of rotation for developed trihedron. Differentiating Eqn. (5.2) and substituting Eqns.(4.8) and (5.8) into the result, we obtain

$$\beta'_{r_1}(s) = \Gamma_{r_1} r_1 + \Delta_{r_1} r_3 \tag{5.10}$$

where

$$\begin{cases} \Gamma_{r_1} = -\mu'_{r_1} \\ \Delta_{r_1} = \frac{\mu\theta' + \Gamma_{k_1}\lambda}{\kappa} \end{cases} \tag{5.11}$$

The first-order angular variation of natural trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \tag{5.12}$$

where κ and $\tau = \left\langle r'_2, r_3 \right\rangle$ are the curvature and torsion of the timelike ruled surface φ, respectively.
To find simpler expressions for the curvature and torsion, we substituting Eqn. (5.6) into Eqn. (5.7) which gives

$$\kappa = \theta' \cos ech\eta \tag{5.13}$$

Differentiating eqn.(5.4) and by using eqns (4.4) and (5.8) we have

$$\tau = -\eta' \tag{5.14}$$

As in the case of the natural trihedron, eqn. (5.16) may also be written as

$$U_{r_2} = \kappa r_3 - \tau r_1 \tag{5.15}$$

which is the Darboux vector of the natural trihedron.

Hence, observe that both the Darboux vector of the natural trihedron and the Darboux vector of the developed trihedron describe the angular motion of the timelike ruled surface and the timelike central normal surface.

6. Relationship Between The Frames

Path of a robot may be represented by a tool center point and tool frame of end-effector. For tool frame, there are two cases. The tool frame is represented by three mutually perpendicular unit vector $[O, A, N]$,

i) where O is the spacelike orientation vector, A is the timelike approach vector and N is the spacelike normal vector, shown in figure 1.

ii) where O is the spacelike orientation vector, A is the spacelike approach vector and N is the timelike normal vector, shown in figure 1.

Each vector of tool frame in end-effector defines its own ruled surface while the robot moves. Let $O = k_1$ and the vector O are called directix and ruling, respectively. Then, ,for $v = 0$,the surface frame $[O, S_n, S_b]$ of timelike ruled surface φ may be determined as follows;

$$S_n = \frac{\varphi_s \times \varphi_v}{\|\varphi_s \times \varphi_v\|} \tag{6.1}$$

which is the unit spacelike normal of timelike ruled surface in TCP.

Substituting Eqns. (4.4), (4.8) and Eqn.(4.6) into Eqn.(6.1) we obtain

$$S_n = \frac{(\Delta_{k_1} + \mu_{k_1}\theta')t_1 + \mu r_1}{\sqrt{(\Delta_{k_1} + \mu_{k_1}\theta')^2 + \mu^2}} \tag{6.2}$$

where Δ_{k_1}, μ_{k_1}, θ' and μ are as defined by Eqs.(4.9), (4.7), (5.3) and (3.3), respectively.

$$S_b = O \times S_n \tag{6.3}$$

is unit timelike binormal vector of the timelike ruled surface.

Substituting Eqn.(6.2) into Eqn.(6.3) gives

$$S_b = \frac{-(\Delta_{k_1} + \mu_{k_1}\theta')r_1 - \mu t_1}{\sqrt{(\Delta_{k_1} + \mu_{k_1}\theta')^2 + \mu^2}} \tag{6.4}$$

The orientation of the surface frame relative to the tool frame and the developed trihedron is shown in figure 1. Let ζ be the angle between spacelike vector and the timelike approach vector . Then, we have

$$\langle S_n, A\rangle = \sinh\zeta \ , \ A \times O = N \tag{6.5}$$

We may express the results in matrix form as

$$\begin{pmatrix} O \\ A \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh\zeta & \cosh\zeta \\ 0 & \cosh\zeta & \sinh\zeta \end{pmatrix} \begin{pmatrix} O \\ S_n \\ S_b \end{pmatrix} \tag{6.6}$$

Let the angle between the vectors S_b and t_1 be σ. Then, we have

$$\begin{cases} S_n = \sinh\sigma r_1 + \cosh\sigma t_1 \\ S_b = \cosh\sigma r_1 + \sinh\eta t_1 \end{cases} \tag{6.7}$$

From Eqns.(6.6) and (6.7) we can write

$$\begin{pmatrix} O \\ A \\ N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\Sigma & \sinh\Sigma \\ 0 & \sinh\Sigma & \cosh\Sigma \end{pmatrix} \begin{pmatrix} k_1 \\ r_1 \\ t_1 \end{pmatrix} \tag{6.8}$$

where $\Sigma = \ \zeta + \sigma$ describes the orientation of the end-effector.

7. Differential Motion Of The Tool Frame

The spacelike curve generated by TCP from eqn. (3.2) is

$$\alpha(s) = \beta(s) + \mu\overline{R}(s) \tag{7.1}$$

Differentiating eqn. (7.1) with respect to the arc length, using eqns.(3.6) and (3.4), the first-order positional variation of the TCP, expressed in the generator trihedron is

$$\alpha'(s) = \left(\Gamma + \mu' R\right) r + \mu t + \Delta k \tag{7.2}$$

Substituting eqn. (4.1) into eqn.(7.2), gives

$$\alpha' = \left(\left(\Gamma + \mu' R\right)\sinh\theta + \Delta\cosh\theta\right)k_1 + \left(\left(\Gamma + \mu' R\right)\cosh\theta + \sinh\theta\right)r_1 + \mu t_1 \tag{7.3}$$

Also, substituting eqn. (6.8) into eqn. (7.3), gives

$$\alpha' = \left(\left(\Gamma + \mu' R\right)\sinh\theta + \Delta\cosh\theta\right)O + \left(\cosh\Sigma\left(\left(\Gamma + \mu' R\right)\cosh\theta + \Delta\sinh\theta\right) + \mu\sinh\Sigma\right)A + \left(\sinh\Sigma\left(\left(\Gamma + \mu' R\right)\cosh\theta + \Delta\sinh\theta\right) + \mu\cosh \tag{7.4}$$

Differentiating eqn. (6.8) and substituting eqn.(4.4) into the result the first order angular variation of the tool frame gives

$$\frac{d}{ds}\begin{pmatrix} O \\ A \\ N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \theta'\cosh\Sigma & \sinh\Sigma\left(\lambda + \Sigma'\right) & \cosh\Sigma\left(\lambda + \Sigma'\right) \\ \theta'\sinh\Sigma & \cosh\Sigma\left(\lambda + \Sigma'\right) & \sinh\Sigma\left(\lambda + \Sigma'\right) \end{pmatrix}\begin{pmatrix} k_1 \\ r_1 \\ t_1 \end{pmatrix} \tag{7.5}$$

The first-order angular variation of tool frame $\{O, A, N\}$ may be expressed in the matrix form as

$$\frac{d}{ds}\begin{pmatrix} O \\ A \\ N \end{pmatrix} = \begin{pmatrix} 0 & -\theta'\sinh\Sigma & \theta'\cosh\Sigma \\ -\theta'\sinh\Sigma & 0 & \lambda + \Sigma' \\ \theta'\cosh\Sigma & \lambda + \Sigma' & 0 \end{pmatrix}\begin{pmatrix} O \\ A \\ N \end{pmatrix} \tag{7.6}$$

As in the case of the tool frame, may also be written as

$$U_A = -\left(\lambda + \Sigma'\right)O + \left(\theta'\cosh\Sigma\right)A - \left(\theta'\sinh\Sigma\right)N \tag{7.7}$$

which is the Darboux vector of the tool frame. Here, we obtain by using eqn.(6.8)

$$U_A = -\left(\lambda + \Sigma'\right)k_1 + \theta' t_1 \tag{7.8}$$

Also, substituting eqn. (4.10) into eqn. (7.8), gives

$$U_A = U_{k_1} - \Sigma' k_1 \tag{7.9}$$

Hence, angular variation of tool frame that according to developed frame is the rotation around the r_1 vector.

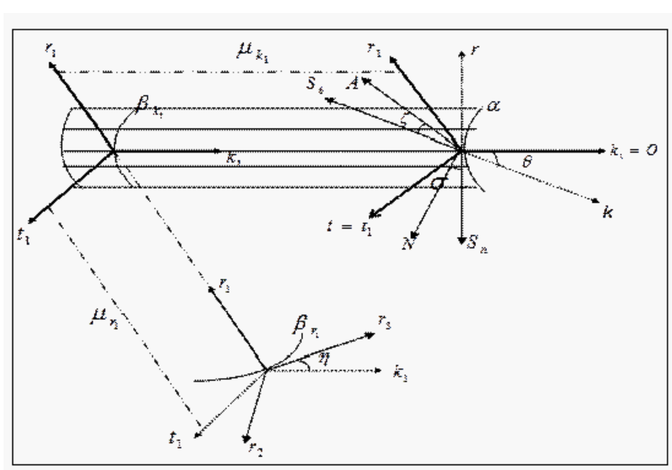

Fig. 1. The relationship between frames.

8. Example

Consider the timelike ruled surface

$$X(s,v) = \left(-\sinh s\left(\sqrt{2}+v\right), s+2v, \cosh s\left(\sqrt{2}+v\right)\right)$$

where $\alpha(s) = \left(-\sqrt{2}\sinh s, s, \sqrt{2}\cosh s\right)$ (timelike) is the base curve $\overline{R}(s) = (-\sinh s, 2, \cosh s)$ (spacelike) is the generator. $-\pi \leq s \leq 0$, $-1 \leq v \leq 1$,(Fig. 2).Since , $\left\|\overline{R}\,'(s)\right\| = 1$ the generator trihedron is

$$\begin{cases} r(s) = \frac{1}{\sqrt{5}}\left(-\sinh s, 2, \cosh s\right), \\ t(s) = \left(\cosh s, 0, \sinh s\right), \\ k(s) = \frac{1}{\sqrt{5}}\left(2\sinh s, 1, -2\cosh s\right). \end{cases}$$

A straight forward computation shows that

$\mu(s) = \sqrt{2}, \Gamma(s) = \frac{2}{\sqrt{5}},\ \Delta(s) = \frac{1}{\sqrt{5}}$ and $\gamma = \left\langle \overline{R}'', \overline{R}' \times \overline{R}\right\rangle = -2.$

Also, the Darboux vector of generator trihedron $U_r = \frac{1}{\sqrt{5}}\left(3\sinh s, 0, -4\cosh s\right)$.

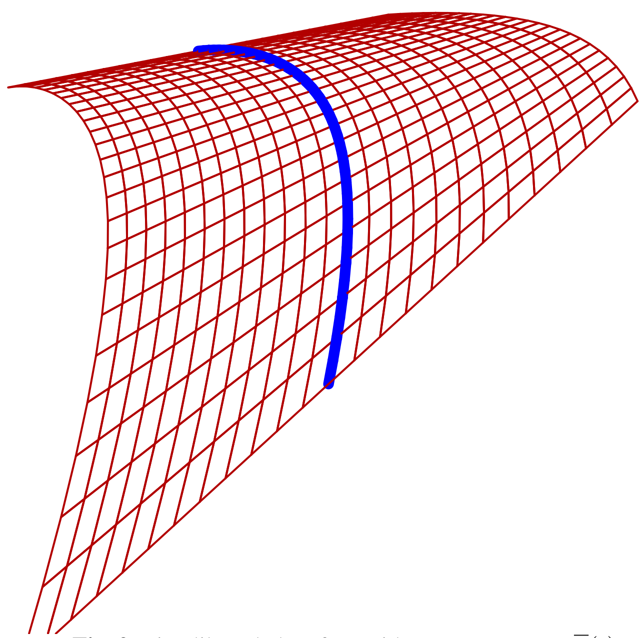

Fig. 2. Timelike ruled surface with generator vector $\overline{R}(s)$

The developed trihedron is defined by,

$$\begin{cases} k_1 = \left(\frac{1}{\sqrt{5}}\sinh s\left(2\cos\theta(s) - \sin\theta(s)\right), \frac{1}{\sqrt{5}}\left(2\sin\theta(s) + \cos\theta(s)\right), \frac{1}{\sqrt{5}}\cosh s\left(\sin\theta(s) - 2\cos\theta(s)\right)\right), \\ r_1 = \left(-\frac{1}{\sqrt{5}}\sinh s\left(\cos\theta(s) - 2\sin\theta(s)\right), \frac{1}{\sqrt{5}}(2\cos\theta(s) - \sin\theta(s)), \frac{1}{\sqrt{5}}\cosh s(\cos\theta(s) + 2\sin\theta(s))\right), \\ t_1 = \left(-\cosh s, 0, \sinh s\right). \end{cases}$$

Therefore, timelike trajectory ruled surface family with a common trajectory curve is defined by

$$\varphi(s,v) = \left(\sinh s(-\sqrt{2} + \frac{v}{\sqrt{5}}(2\cos\theta(s) - \sin\theta(s))), s + \frac{v}{\sqrt{5}}(2\sin\theta(s) + \cos\theta(s))), \cosh s(\sqrt{2} + \frac{v}{\sqrt{5}}(\sin\theta(s) - 2\cos\theta(s)))\right) \tag{8.1}$$

If we take $\theta(s) = s,\ -\pi \leq s \leq 0$ and $-1 \leq v \leq 1$ then we obtain $\varphi_1 = \varphi_1(s,v)$ a member of the timelike trajectory ruled surface family with a common trajectory curve in the developed trihedron as shown in (Fig.3) in green.

$$\varphi_1(s,v) = \left(\sinh s(-\sqrt{2}+\frac{v}{\sqrt{5}}(2\cos s-\sin s)), s+\frac{v}{\sqrt{5}}(2\sin s+\cos s)), \cosh s(\sqrt{2}+\frac{v}{\sqrt{5}}(\sin s-2\cos s))\right) \quad (8.2)$$

If we take $\theta(s)=s^2$, $-\pi \leq s \leq 0$ and $-1 \leq v \leq 1$ then we obtain $\varphi_2=\varphi_2(s,v)$ another member of the timelike trajectory ruled surface family with a common trajectory curve in the developed trihedron as shown in (Fig.3) in red.

$$\varphi_2(s,v) = \left(\sinh s(-\sqrt{2}+\frac{v}{\sqrt{5}}(2\cos(s^2)-\sin(s^2))), s+\frac{v}{\sqrt{5}}(2\sin(s^2)+\cos(s^2))), \cosh s(\sqrt{2}+\frac{v}{\sqrt{5}}(\sin(s^2)-2\cos(s^2)))\right) \quad (8.3)$$

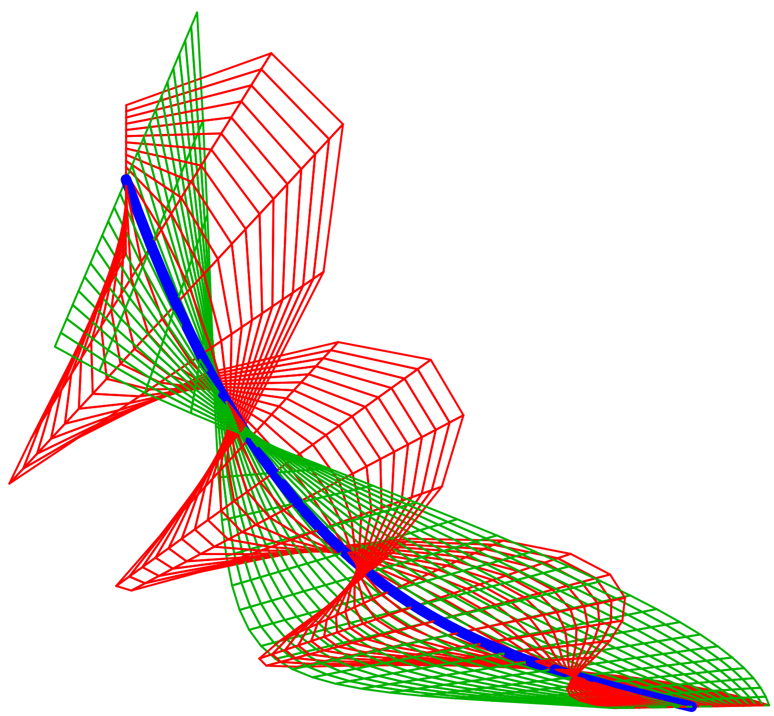

Fig. 3. Timelike trajectory ruled surface.

9. Conclusion

In the 3-dimensional Minkowski space, based on the theory of the curve of the ruled surface, the improved robot end-effector motion of the spacelike ruling timelike ruled surface was examined by investigating its differential properties. In addition, the spacelike ruling timelike ruled surface was observed to be the robot trajectory surface timelike ruled surface. It was also observed that the derivative formulas of the tool frame used to define the trajectory of the robot depend on the Lancret curvature of the given surface, the angular velocity of the Darboux angle and the angle of rotation between the frames. The robot end effector motion is developed by using the Lancret curvature of the surface with the rotation trihedron. Finally, we obtained trajectory ruled surface family with a common trajectory curve with the developed frame. Therefore, we changed trajectories of the robot end effector motion according to the Darboux angle function. Also, the motion of robot end effector is illustrated with examples by two of the desired trajectory of the robot end effector motion.

References

[1] A. Turgut and H. H. Hacısalihoğlu, “Timelike ruled surfaces in the Minkowski 3-space,” Far East Journal of Mathematical Sciences, vol. 5, no. 1, pp. 83–90, 1997.

[2] A. Turgut and H. H. Hacısalihoğlu, "Timelike ruled surfaces in the Minkowski 3-space. II," Turkish Journal of Mathematics, vol. 22, no. 1, pp. 33–46, 1998.

[3] A. O. Öğrenmis, H. Balgetir, and M. Ergüt, "On the ruled surfaces in Minkowski 3-space ," Journal of Zhejiang University: Science A, vol. 7, no. 3, pp. 326–329, 2006.

[4] B. S. Ryuh, Robot trajectory planing using the curvature theory of ruled surfaces, Doctoral dissertion, Purdue Universty, West Lafayette, Ind, USA, 1989.

[5] McCarthy, J. M. and Roth, B., The curvature theory of line trajectories in spatial kinematics, ASME Journal of Mechanical Design, 103 (1981), No.4, 718-724

[6] B. S. Ryuh and G. R. Pennock, 'Accurate motion of a robot end-effector using the curvature theory of ruled surfaces, Journal of mechanisms, Transmissions, and Automation in Design, vol. 110, no. 4, pp. 383-388, 1988.

[7] B.S.Ryuh, K.M.Lee and M.J.Moon , A Study on the Dual Curvature Theory of a Ruled Surface for the Precision Control of a Robot Trajectory , A Scientific and Technical Publishing Company, 2006.

[8] C.H.Chu* , W.N.Huang* , Y.Y.Hu, Machining accuracy improvement in five-axis flank milling of ruled surfaces, Volume 48, pp. 914-921,2008.

[9] J. H. Kim, B.S.Ryuh and G.R. Pennock, Development of a trajectory generation method for a five-axis NC machine, Mechanism and Machine Theory 36 (2001) 983-996.

[10] A.Gasparetto and V.Zanotto, A New Method for Smooth Trajectory Planning of Robot Manipulators, Mechanism and Machine Theory 42(2007)455-471.

[11] F.L.Litvin and X.C.Gao, "Analytical representation of trajectory of manipulators, trends and developments in mechanisms, machines, and robotics," in the ASME Design Technology Conferences, the 20th Biennial Mechanisms Conference, vol.15-3, pp.481-485, Kissimmee, Fla, USA, September 1988.

[12] R. Paul, Manipulator Cartesian path control, IEEE Transactions on Systems, Man and Cybernetics, vol.9, no. 11, pp.702-711, 1979.

[13] C. Ekici, Y. Ünlütürk, M.Dede and B.S.Ryuh, On Motion of Robot End-Effector Using the Curvature Theory of Timelike Ruled Surfaces with Timelike Rulings, Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2008, Article ID 362783, 19 pages doi:10.1115/2008/362783.

[14] T. Turhan and N.Ayyıldız, On Curvature Theory of Ruled Surfaces with Lightlike Ruling in Minkowski 3-Space, Int. Journal of Mathematical Sciences and Applications, Vol.1,No.3, September 2011.

[15] Sahiner B., Kazaz M., and Ugurlu H.H., A Study on Motion of a Robot End-Effector Using the Curvature Theory of Dual Unit Hyperbolic Spherical Curves , Filomat 30:3 (2016), 791–802. DOI 10.2298/FIL1603791S.

[16] O'Neill, B., Semi-Riemannian Geometry With Applications to Relativity, Academic Press, O-12-526740-1, 1983.

[17] Ratcliffe, J.G.: Foundations of Hyperbolic Manifolds, Springer-Verlag, 0-387-33197-2, 1994.

[18] Walrave J., Curves and surfaces in Minkowski space, PhD. Thesis, K. U. Leuven Faculteit Der Wetenschappen, 1995.